Fractal clustering of inertial particles in random flows

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It is shown that preferential concentrations of inertial (finite-size) particle suspensions in turbulent flows follow from the dissipative nature of their dynamics. In phase space, particle trajectories converge toward a dynamical fractal attractor. Below a critical Stokes number (non-dimensional viscous friction time), the projection on position space is a dynamical fractal cluster; above this number, particles are space filling. Numerical simulations and semi-heuristic theory illustrating such effects are presented for a simple model of inertial particle dynamics.

Dust particles, droplets, bubbles and various impurities advected by turbulent flow have usually a finite size and a mass density differing from that of the carrier fluid. Contrary to passive tracers, whose dynamics is conservative (when the flow is incompressible), the dynamics of such *inertial particles* is rendered dissipative by the Stokes drag, which can lead to strongly inhomogeneous spatial distributions. The full statistical description of such preferential concentrations is still an open question with many natural and industrial applications, such as the growth of rain drops in subtropical clouds,[1] the formation of planetesimals in the early Solar system,[2] optimization of combustion processes and the coexistence problems between several species of plankton.[3]

Maxey and Riley[4] derived an equation for the motion of a rigid spherical particle embedded in an incompressible flow. They assume that (i) the particle is smaller than the smallest turbulent scale of the carrier flow and that (ii) the Reynolds number associated to its size and its relative velocity with respect to the fluid is sufficiently small to approximate the surrounding flow by a Stokes flow. Then, the forces exerted on the particle are buoyancy, the force due to the undisturbed flow, the Stokes viscous drag, the added mass effect and the Basset–Boussinesq history force. Because of the complexity of the resulting equation of motion, simpler models are generally used. For instance, when the Stokes drag is very strong, the dynamics is close to that of passive tracer particles and the discrepancy can be captured by a spatial Taylor expansion, leading to a model in which the particles are advected by a synthetic flow comprising a small compressible component.[5,6] What singles out the model proposed here is its ability to take into account the full phase-space dynamics of the particles and to capture the essential features of their dissipative motion. We are interested in the "Batchelor régime" of the particles, meaning that we focus on spatial scales smaller than the Kolmogorov dissipation scale η. After rescaling of space, time and velocity respectively by factors η, η^2/ν and ν/η, and assuming that the particle velocity is sufficiently close to that of the fluid, the Newton equation satisfied by its trajectory $\mathbf{X}(t)$ reduces to

$$\ddot{\mathbf{X}} = \beta \frac{d}{dt}\left[\boldsymbol{u}(\mathbf{X},t)\right] - \frac{1}{\mathrm{S}_\eta}\left[\dot{\mathbf{X}} - \boldsymbol{u}(\mathbf{X},t)\right], \qquad (1)$$

where $\beta \equiv 3\rho_f/(\rho_f + 2\rho_p)$ is the added-mass factor (ρ_f and ρ_p are the fluid and the particle mass densities) and $\mathrm{S}_\eta \equiv a^2/(3\beta\eta^2)$ is the *Stokes number* associated to the dissipative scale of the carrier flow (a being the particle radius). Introducing the *co-velocity* $\mathbf{V} \equiv \dot{\mathbf{X}} - \beta\boldsymbol{u}(\mathbf{X},t)$, the equation of motion can be interpreted in terms of the $(2\times d)$-dimensional dynamical system

$$\dot{\mathbf{X}} = \beta\boldsymbol{u}(\mathbf{X},t) + \mathbf{V}, \qquad (2)$$

$$\dot{\mathbf{V}} = \frac{1}{\mathrm{S}_\eta}\left[(1-\beta)\boldsymbol{u}(\mathbf{X},t) - \mathbf{V}\right]. \qquad (3)$$

The motion of the particles is clearly dissipative, even if the carrier flow is itself incompressible: indeed, when $\nabla\cdot\boldsymbol{u} = 0$, the contraction rate in phase space reduces to $-d/\mathrm{S}_\eta$ and is strictly negative, inducing a uniform contraction. As a consequence, the long-time dynamics of the particles is characterized by the presence of an attractor, that is a dynamical fractal set of the phase space toward which the trajectories $(\mathbf{X}(t),\mathbf{V}(t))$ converge. Important information on the dynamical system (2)-(3), regarding stability, Lyapunov exponents, etc is obtained from the linearized equation governing the separation $\mathbf{R}(t) \equiv (\delta\mathbf{X}(t), \delta\mathbf{V}(t))$ between two infinitesimally close trajectories of the phase space. For scales within the viscous scale of turbulence, the velocity field $\boldsymbol{u}$ can be considered spatially smooth and the separation $\mathbf{R}(t)$ obeys the linear differential equation

$$\dot{\mathbf{R}} = \mathcal{M}_t\,\mathbf{R}, \qquad \mathcal{M}_t \equiv \begin{bmatrix} \beta\boldsymbol{\sigma}(t) & \mathcal{I}_d \\ \dfrac{1-\beta}{\mathrm{S}_\eta}\boldsymbol{\sigma}(t) & \dfrac{1}{\mathrm{S}_\eta}\mathcal{I}_d \end{bmatrix}, \qquad (4)$$

where $\boldsymbol{\sigma}$ is the strain matrix of the carrier flow along the path of a reference particle: $\sigma_{ij}(t) \equiv \partial_j u_i(\boldsymbol{X}(t),t)$, and $\mathcal{I}_d$ is the d-dimensional identity matrix. A full stability analysis of the dynamics can easily be done;[7] it relates the eigenvalues of the evolution matrix $\mathcal{M}_t$ to that of the stress tensor $\boldsymbol{\sigma}$ representative of the local structure of the carrier flow. In both two and three dimensions, this leads to distinguishing between heavy ($\beta < 1$) and light particles ($\beta > 1$): the former are usually ejected from the elliptic regions, while the latter may cluster there in a pointwise manner. We therefore expect the vortex cores to be regions of high concentration of light particles and of low concentration of heavy particles, feature which is

generally observed in experiments and simulations (for a review, see Eaton and Fessler[8]).

We focus here on suspensions of particles with a volume fraction sufficiently small to neglect their interactions and collisions. Typically, the phase-space attractor on which the particles concentrate is a fractal object which may be characterized by various dimensions, in particular a non-random Hausdorff dimension d_H. As the position of the particles is obtained by projection from the $2d$-dimensional phase space onto the d-dimensional physical space, the convergence to the attractor is responsible for strong inhomogeneities in the large-times distribution of particles. More precisely, a standard result of the geometrical theory of fractal sets[9] states that if $d_H < d$, the distribution of particles in the physical space is itself a fractal set with Hausdorff dimension d_H, whereas if $d_H > d$, the particles fill the whole space. Hence, depending on the value of the dimension d_H of the attractor, two different régimes are distinguished. Clearly, the dimension of the attractor is a function of the Stokes number S_η and of the added mass parameter β, and generally also depends on the statistical properties of the velocity of the carrier fluid. Leaving aside this latter dependence, let us first note what can be easily inferred on the behavior of d_H as a function of S_η and β. First, in the limit of vanishing Stokes numbers, there is a reduction of dimensionality and the dynamics of simple tracers is recovered. An initially uniform distribution of particles remains uniform and we hence have $d_H \to d$. Next, for very large Stokes numbers, the particles are less and less influenced by the carrier fluid and their motion becomes ballistic. They thus fill the whole phase space and we have $d_H \to 2d$. In between these two asymptotic régimes, although it is not obvious that the dimension d_H can fall below d, we shall actually show that there is a whole range of Stokes numbers for which $d_H < d$ and, thus, preferential concentration on fractal clusters occurs in physical space.

Finding theoretically or numerically the Hausdorff dimension d_H of the attractor is not, in general, a simple task: its determination demands a full understanding of the global dynamics and its numerical measurement requires very large numbers of particles. To obtain a simple estimate of the attractor dimension, Kaplan and Yorke[10] proposed to use the *Lyapunov dimension*. It is given by the Lyapunov exponents $\lambda_1 > \ldots > \lambda_{2d}$ which measure the exponential growth of infinitesimal distances, surfaces and volumes in the phase space and are expressed in terms of the limiting singular values of the Jacobi matrix, i.e.

$$\lambda_j \equiv \lim_{t\to\infty} \frac{1}{t} \ln |\mathcal{J}_t \boldsymbol{e}_j| , \quad \mathcal{J}_t \equiv \mathcal{T}\mathrm{exp} \int_0^t \mathcal{M}_s ds, \tag{5}$$

where the $\boldsymbol{e}_j$'s are the eigenvectors of the symmetric matrix $\mathcal{J}_t^{\mathrm{T}} \mathcal{J}_t$ and $\mathcal{T}\mathrm{exp}$ denotes the time-ordered exponential. The Lyapunov dimension is defined as

$$d_L \equiv j - \frac{\lambda_1 + \ldots + \lambda_j}{\lambda_{j+1}}, \tag{6}$$

where j satisfies $\lambda_1 + \ldots + \lambda_j \geq 0$ and $\lambda_1 + \ldots + \lambda_{j+1} < 0$. Beside being a simple estimate of the dimension of the attractor, it was actually shown[11] that the Lyapunov dimension gives a rigorous upper bound for the Hausdorff dimension d_H of the attractor.

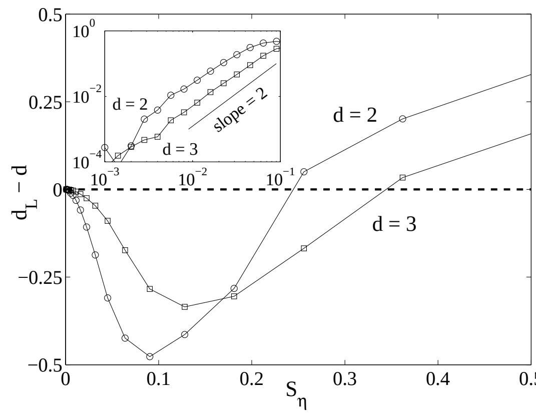
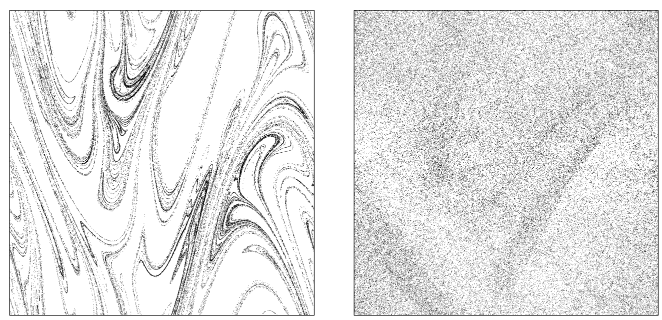


FIG. 1: Difference between the Lyapunov dimension d_L of heavy particles ($\beta = 0$) and the physical space dimension d, versus the Stokes number S_η (circle: $d = 2$, square: $d = 3$). The critical Stokes number corresponds to the value for which $d_L = d$. Upper-left inset : same in log-log coordinates showing a quadratic behavior at small Stokes numbers.



(a) $\mathrm{S}_\eta = 10^{-2}$ (b) $\mathrm{S}_\eta = 1$

FIG. 2: Snapshots of the position of $N = 10^5$ heavy particles ($\beta = 0$) associated to two different Stokes numbers: (a) smaller than the critical value, for which the particles form fractal clusters, and (b) larger than the critical value for which they fill the whole domain. The carrier incompressible flow is generated randomly by 4 independent modes with a finite correlation time.

We performed numerical experiments in two and three dimensions for a space-periodic carrier flow generated randomly by the superposition of few independent Gaussian Fourier modes with a correlation time of the order of unity (this specific form for the carrier flow was considered by Sigurgeirsson and Stuart[12] who proved the existence of a random dynamical attractor). The Lyapunov exponents are calculated by the use of the standard technique of Benettin *et al.*[13], and the resulting Lyapunov dimension is represented both for $d = 2$ and $d = 3$ in Fig. 1 for the case of very heavy particles ($\beta = 0$). Two important observations can be made from these simula-

tions. First, fractal clustering of particles already occurs at very small Stokes number where the Lyapunov dimension behaves as $d_L \simeq d - C\mathrm{S}_\eta^2$ with $C > 0$. This quadratic behavior near zero was predicted for the correlation dimension by Balkovsky *et al.*[6] using the method of the synthetic compressible flow cited earlier, as an approximation at low Stokes numbers. The second observation is the presence of a critical value for the Stokes number below which the attractor dimension is smaller than d and where the particles form fractal clusters in the physical space. The two régimes corresponding to different values of the Stokes number are illustrated for $d = 2$ in Fig. 2. When the Stokes number is below the critical value (a), the particles concentrate onto a fractal set and both very dense and almost empty regions appear. On the contrary, for a Stokes number above the threshold (b), the particles fill the whole domain, albeit with a non uniform density.

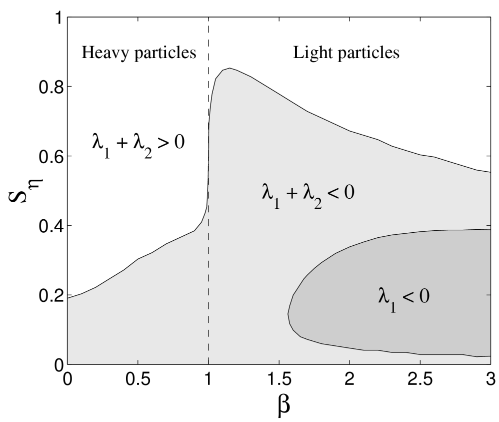


FIG. 3: Phase diagram in the parameter space (β, S_η) for the two-dimensional case, representing the three different régimes classified by the behavior of the Lyapunov exponents.

When the particles have a finite mass ($\beta \neq 0$), there also exists a critical Stokes number for their concentration onto fractal clusters. Figure 3 shows for $d = 2$ the phase diagram obtained numerically which divides the parameter space (β, S_η) between three different régimes. When the sum of the two largest Lyapunov exponents is negative (light-gray domain), we have $d_L < d$ and the particles form fractal clusters in the physical space. When the sum is positive (white domain), we have $d_L > d$ and the particles fill the whole domain. The third case occurs only for particles lighter than the fluid and corresponds to a negative largest Lyapunov exponent (dark gray area). The particles form pointwise clusters and, when the domain is bounded, they all converge to a single trajectory.

We now present a heuristic argument, which we already gave in the case $\beta = 0$ of heavy particles,[14] and which predicts the threshold in Stokes number. It relies on the use of the *stability exponents*, that are the exponential growth rates of the eigenvalues of the Jacobi matrix $\mathcal{J}_t$ defined in (5). Using Browne's theorem which bounds the singular values of a square matrix by its eigenvalues, a necessary condition for the fractal clustering of particles is that the sum of the d largest stability exponents is positive. For heavy particles ($\beta < 1$), this sum can be estimated from the local analysis of the dynamics. First, since such particles tend to cluster within the hyperbolic regions of the flow, they are spending there a fraction of time much larger than in the elliptic regions. Let us assume that the relationship between the eigenvalues of the evolution matrix $\mathcal{M}_t$ and those of the stress tensor $\boldsymbol{\sigma}(t)$ can be extended to the stability exponents, at least as an approximation; we can then derive a necessary condition for the presence of fractal clusters in the physical space. For $d = 2$, this condition can be easily written as

$$\mathrm{S}_\eta \leq \frac{1}{\lambda_f \beta^2} \left(\beta - 2 + 2\sqrt{1 - \beta + \beta^2} \right), \tag{7}$$

where λ_f is the (non-dimensional) largest Lyapunov exponent associated to the carrier velocity field and calculated along the trajectory of a simple fluid particle. It was easily verified that this bound is compatible with what is observed numerically in Fig. 3.

It is often stated in the literature that the clustering of inertial particles is essentially due to the presence of coherent structures in turbulence (see., e.g., Squires and Eaton[15]). It is indeed generally assumed that the structures with long life times appearing in the flow are responsible for a deterministic motion of the particle leading to their concentration inside or outside the vortices. Although this argument, based on the local structure of the carrier flow, allowed us to find an upper bound for the critical value of the Stokes number, it is important to stress that preferential concentrations of particles arise solely as a consequence of the dissipative character of the motion. Indeed, we have also performed simulations with carrier flows which are delta-correlated in time and that are thus completely devoid of structure.[7] These simulations show that the dependence of the Lyapunov dimension on the Stokes number S_η is very similar to that obtained in Fig. 1. The main difference is the behavior of d_L as $\mathrm{S}_\eta \to 0$: in the delta-correlated case, the Lyapunov dimension tends linearly, and not quadratically, to the space dimension d. In the delta-correlated case, preferential concentrations are actually stronger than for a finite correlation time, contradicting the mechanism frequently invoked to explain concentrations.

Of course, it is not sufficient to know that the particles are concentrated in fractal objects; a fuller statistical description of their distribution is desirable. In particular, for a quantitative description of their spatial intermittency, one needs their multifractal properties (scaling exponents of the various moments of the mass contained in a sphere of radius r; see e.g. Chap. 8 of the book by Frisch[16]). Preliminary results, for heavy particles, indicate that strong spatial intermittency can be present, even when the Lyapunov dimension is just slightly below that of the physical space. Let us also mention that more quantitative results are likely to be within reach using

multi-time methods in the asymptotics $S_\eta \ll 1$. Small Stokes numbers are of considerable interest since in most natural and industrial situations, this is the case. A last remark concerns the extension of this approach to the case of real turbulent flows. To confirm the existence of a threshold for fractal clustering of inertial particles, one needs to resolve scales which are much smaller than the Kolmogorov dissipation scale η. This is quite a challenge, both for laboratory and numerical experiments.

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